An Even Simpler "Truly Elementary" Proof of Bertrand's Theorem

Jared Galbraith [a] and Jacob Williams [b]

Southern Nazarene University
6729 NW 39th Expressway, Bethany, OK 73008

[a]Corresponding author: jargalbrai@gmail.com
[b] jacwilli289@gmail.com

Abstract: We present a further simplified derivation of a "truly elementary" proof of Bertrand's theorem, which predicts the exponents in central power-law potentials that produce closed orbits.

INTRODUCTION

Bertrand's theorem[1] proves that for a central force power-law potential energy $V(r) \sim r^n$, closed orbits exist only for $n = -1$ and $+2$. An elegant "truly elementary" proof of the theorem was recently published by S. Chin.[2] Here we streamline the theorem's proof further, making it even more elementary.

We review criteria for an orbit to be closed, outline a strategy for determining which values of n give a closed orbit, then consider cases of negative and positive n. To make this note self-contained, we develop an argument along the lines of Chin's, but indicate where we introduce an additional simplification. For comparison, Chin's argument we replace is presented in the Appendix.

CLOSED ORBITS: CRITERIA, STRATEGY, AND CASES

Closed Orbit Criteria and Strategy

In central force motion, the force and potential energy depend only on the distance between the force center and the particle, suggesting the use of spherical coordinates (r, θ, φ). Because angular momentum is conserved, the orbit may be mapped in the $\theta = \pi/2$ plane, and the trajectory specified as $r = r(\varphi)$. In a closed orbit, let r_2 be the maximum and r_1 be the minimum values of r. The angle φ_A between them, the *apsidal angle*, is one-half the spatial angular period for the radial oscillation $r_2 \rightarrow r_1 \rightarrow r_2$ (Fig. 1).

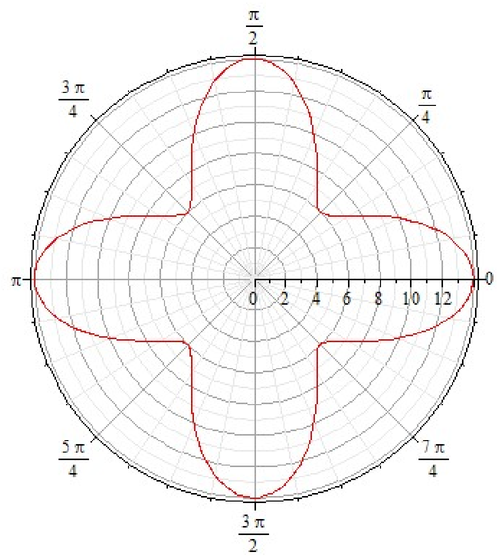

FIGURE 1. For the curve (red) the apsidal angle φA is $\pi/4$.

For an orbit to close in an integral number M revolutions so that $r(\varphi + 2\pi M) = r(\varphi)$, an integral number N periods of the radial oscillation must fit into $2\pi M$. Thus $2\varphi_A N = 2\pi M$, or

$$\varphi_A = \pi/R \tag{1}$$

where R is a rational number.

An effective way to predict a particle's orbit in a central potential employs the conservation of energy and angular momentum.[3] Since the particle of reduced mass m moves with velocity

$$\boldsymbol{v} = \dot{r}\hat{\boldsymbol{r}} + (r\dot{\varphi})\hat{\boldsymbol{\varphi}} \tag{2}$$

(overdots denote time derivatives), the angular momentum is

$$\boldsymbol{L} = (mr^2\dot{\varphi})\hat{\boldsymbol{z}}. \tag{3}$$

Using Eqs. (2) and (3), the mechanical energy E is

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + V(r). \tag{4}$$

The $L^2/2mr^2$ contribution to the kinetic energy behaves mathematically like a repulsive $1/r^2$ potential energy; it is sometimes called the "centrifugal potential." Together with the potential energy $V(r)$ they make the *effective potential* $V_e(r)$:

$$V_e(r) \equiv \frac{L^2}{2mr^2} + V(r). \tag{5}$$

Solving Eq. (5) for $\dot{r} = \frac{dr}{d\varphi}\dot{\varphi}$, again using Eq. (3) and introducing

$$u = 1/r, \tag{6}$$

Eq. (5) yields an integration for $\varphi = \varphi(r)$,

$$\varphi(r) = \pm\sqrt{\beta}\int \frac{du}{\sqrt{E - \beta u^2 - V(u^{-1})}} \qquad (7)$$

where $\beta \equiv L^2/2m$. After integrating, $\varphi = \varphi(r)$ is inverted to obtain $r = r(\varphi)$, and closure (or not) of the orbit may be judged directly by seeing whether $r(\varphi + 2\pi M) = r(\varphi)$ for integer M. For $n = -1$ (planetary orbits or Rutherford scattering), inverting $\varphi(r)$ produces a conic section $\alpha/r = 1 + \epsilon \cos \varphi$. For $n = 2$ (mass on a radial spring), inverting $\varphi(r)$ gives $(\alpha/r)^2 = 1 - \sin(2\varphi)$. Clearly, an elegant proof of Bertrand's theorem would be straightforward if the antiderivative of the integrand in Eq. (7) presented itself as a function of n for any potential of the form $V(u^{-1}) \sim u^{-n}$. Unfortunately, $\varphi(r)$ as a function of arbitrary n is not forthcoming. Another approach must be attempted.

Solving Eq. (4) for $\dot{r} = \frac{dr}{d\varphi}\dot{\varphi}$ and using Eq. (3) to replace angular velocity with angular momentum, so that $\frac{dr}{d\varphi}\dot{\varphi} = \frac{dr}{d\varphi}\frac{L}{mr^2}$, leads to

$$\frac{L^2}{2mr^4}\left(\frac{dr}{d\varphi}\right)^2 = E - V_e(r). \qquad (8)$$

Recalling $u = 1/r$ and, following Chin, defining

$$L^2/m \equiv m^*, \qquad (9)$$

Eq. (8) may be recast as

$$E = \frac{1}{2}m^*\left(\frac{du}{d\varphi}\right)^2 + \frac{1}{2}m^* u^2 + V(u^{-1}). \qquad (10)$$

The last two terms are the effective potential in terms of u,

$$V_e(r) = V_e(u^{-1}) \equiv \frac{1}{2}m^* u^2 + V(u^{-1}). \qquad (11)$$

In u-space Eq. (10) has the same mathematical form as the kinetic energy plus potential energy of a simple harmonic oscillator—plus a perturbation, $V(u^{-1})$. If $V(u^{-1})$, like $\frac{1}{2}m^* u^2$also happens to be quadratic in u, then the entire $V_e(r)$ is quadratic in u takes the form

$$V_e(r) = \frac{1}{2}\gamma u^2 \qquad (12)$$

for some constant γ. Should that occur, then $u \sim \cos(\omega\varphi)$ where $\omega^2 = \gamma/m^*$. The criteria for a closed orbit, Eq. (1), becomes

$$\varphi_A = \frac{\pi}{\omega} \qquad (13)$$

where, according to Eq. (1), ω must be a rational number. Of course, $V(r)$ is not always quadratic in u. But if a Taylor series expansion of the effective potential is dominated by the quadratic term, then the argument about $\gamma = m^*\omega^2$ holds.

As noted, the potential V can be seen as a perturbation. Since we are dealing with *bound* orbits, closed or not, let us suppose the system that V perturbs is a circular orbit of radius $r_o = 1/u_o$. The effective potential therefore has a minimum at this radius (see Fig. 2). Let us expand the effective potential $V_e(u)$ in a Taylor series about $u = u_o$:

$$V_e(u) = V_e(u_o) + (u - u_o)\left[\frac{dV_e}{du}\right]_{u_o} + \frac{1}{2!}(u - u_o)^2\left[\frac{d^2V_e}{du^2}\right]_{u_o} + \cdots . \qquad (14)$$

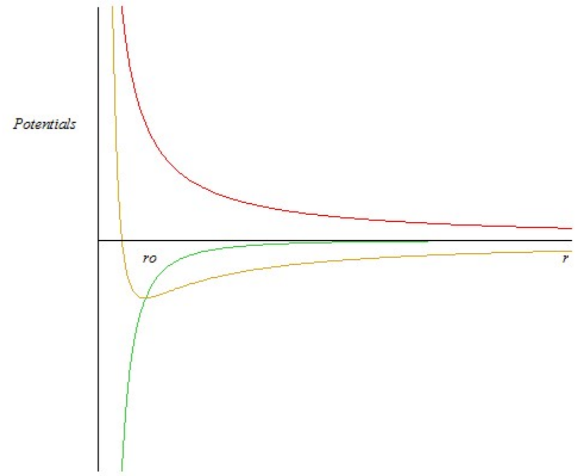

FIGURE 2. The $1/r^2$ (top red) curve is the angular momentum's contribution to $V_e(r)$; the bottom (green) curve illustrates an attractive potential, in this instance $V(r) \sim -1/r$; and the curve (yellow) with the minimum at r_o represents the effective potential V_e(r).

With $V_e(u)$ a minimum at u_o, the first derivative term vanishes. Denote $V_e(u_o) \equiv E_o$, $u - u_o \equiv \varepsilon$, and $\left[\frac{d^2V_e}{du^2}\right]_{u_o} \equiv \Gamma$. The Taylor series may be written

$$V_e(u) = E_o + \frac{1}{2}\varepsilon^2\Gamma + \cdots . \tag{15}$$

Now Eq. (10) may be restated

$$E - E_o = \frac{1}{2}m^*\left(\frac{d\varepsilon}{d\varphi}\right)^2 + \frac{1}{2}\Gamma\varepsilon^2 + \cdots. \tag{16}$$

By examining Γ, which depends on the second derivative of V_e, let us see what constraints it imposes on power-law potentials in producing closed orbits. The first derivative of V_e with respect to u is, in terms of $V(r)$,

$$\frac{dV_e}{du} = m^*u + \frac{dV}{dr}\frac{dr}{du} = m^*u - \frac{1}{u^2}\frac{dV}{dr}; \tag{17}$$

and thus the second derivative becomes

$$\frac{d^2V_e}{du^2} = m^* + \frac{2}{u^3}\frac{dV}{dr} + \frac{1}{u^4}\frac{d^2V}{dr^2}. \tag{18}$$

At $u = u_o$ we obtain

$$\frac{\Gamma}{m^*} = \frac{3V'(r_o)+rV''(r_o)}{V'(r_o)}. \tag{19}$$

For this Γ to be the $\gamma = m^*\omega^2$ of Eq. (12), Γ/m^* must be a positive real number. The question now becomes, what potentials $V(r)$ allow this to happen? Define the function

$$f(r) = \frac{3V'(r)+rV''(r)}{V'(r)}. \tag{20}$$

Since $f(r_o) = \Gamma/m^* = const. > 0$. Therefore for $r \approx r_o$ we can say that $f(r) \approx C = const. > 0$. Then Eq, (20) gives

$$3V' + rV'' = CV'. \tag{21}$$

or

$$(C-3)V' = r\frac{dV}{dr} \tag{22}$$

which integrates to

$$\ln V' = (C - 3)\ln r + \ln k \tag{23}$$

or $dV/dr = kr^{C-3}$ where $k = const.$ Letting $n = C - 2$, a second integration yields

$$V(r) = \frac{k}{n} r^n. \tag{24}$$

Recalling that $\Gamma/m^* = C = n + 2$, and assuming that further terms in the Taylor series may be neglected, we have our simple harmonic oscillator's angular frequency,

$$\omega = \sqrt{\frac{\Gamma}{m^*}} = \sqrt{n+2}. \tag{25}$$

The criteria for the orbit to be closed, Eq. (1), requires $\sqrt{n+2}$ to be a rational number. Clearly $n = -1$ and $n = +2$ make ω a rational number, but what other values of n might produce closed orbits? Why not $n = 7$ or 23 or 34? Even though these choices make $\sqrt{n+2}$ an integer, evidently the rationality of $\sqrt{n+2}$ is a *necessary* but not *sufficient* condition for the orbit to be closed, because the potential $V(r) \sim r^n$ is also constrained by Newtonian mechanics. To find values of n that work, let us divide the real numbers into two groups, $n < 0$, and $n > 0$, and see how the principles of *mechanics* constrain the values of n that make $\sqrt{n+2}$ rational.

The $n < 0$ Case

For $n < 0$, let $n = -s$ with $s > 0$ (note that Eq. (25) requires $-2 \leq n < 0$). Then $V(r) = (k/n)r^n = -(k/s)r^{-s} = -(k/s)u^s$, and Eq.(10) takes the form

$$E = \frac{1}{2}m^*\left(\frac{du}{d\varphi}\right)^2 + \frac{1}{2}m^*u^2 - \frac{k}{s}u^s \tag{26}$$

or

$$Eu^{-s} = \frac{1}{2}m^*u^{-s}\left(\frac{du}{d\varphi}\right)^2 + \frac{1}{2}m^*u^{2-s} - \frac{k}{s}. \tag{27}$$

With the change of variable $x = u^{2-s}$, Eq. (27) says

$$Eu^{-s} + \frac{k}{s} = \frac{1}{2}m^*\left(\frac{2}{2-s}\right)^2\left(\frac{dx}{d\varphi}\right)^2 + \frac{1}{2}m^*x^2. \tag{28}$$

For an orbit to be *bound* with an inverse power-law potential requires $E < 0$. Since the power-law exponent does not depend on the energy, let $E \rightarrow 0^-$, when the particle become barely bound. Then Eq. (28) reduces to

$$\frac{k}{s} = \frac{1}{2}m^*\left(\frac{2}{2-s}\right)^2\left(\frac{dx}{d\varphi}\right)^2 + \frac{1}{2}m^*x^2, \tag{29}$$

which is mathematically identical to the expression for the energy of a simple harmonic oscillator of total energy k/s, mass $4m^*/(2-s)^2$ and spring constant m^*. It therefore has the angular frequency

$$\omega_o = \sqrt{\frac{m^*}{m^*\left(\frac{2}{2-s}\right)^2}} = \frac{2-s}{2}\ . \tag{30}$$

In a simple harmonic oscillator's motion, the coordinate may be positive *or* negative, oscillating with period $T_o = 2\pi/\omega_o$ about the origin. But since $x = u^{2-s} = 1/r^{2-s}$ and $r > 0$, in the graph of the "potential energy" $\frac{1}{2}m^*x^2$, the "motion" can take place only on the $x > 0$ side of the parabola. Therefore the period is $T = \frac{1}{2}T_o$, so that $\omega = 2\omega_o$. The condition for a closed orbit, Eq. (1), now says

$$\varphi_A = \frac{\pi}{2\omega_o} = \frac{\pi}{2-s} = \frac{\pi}{2+n}. \quad (31)$$

But we also require, from Eq. (25),

$$\varphi_A = \frac{\pi}{\sqrt{2+n}}. \quad (32)$$

Agreement between both expressions for φ_A requires $2 + n = \sqrt{2+n}$ and thus $n = -1$. The *only* closed orbit that results when $n < 0$ is $n = -1$.

The *n* > 0 Case

Turning to $n > 0$, Eq. (10) becomes

$$E = \frac{1}{2}m^* \left(\frac{du}{d\varphi}\right)^2 + \frac{1}{2}m^* u^2 + \frac{k}{n}\frac{1}{u^n}. \quad (33)$$

Following the same procedure as in the $n < 0$ case, we multiply Eq. (33) by u^n then let $x^2 \equiv u^{2+n}$. In this way Eq. (33) is recast as

$$\frac{E}{r^n} - \frac{k}{n} = \frac{1}{2}m^* \left(\frac{2}{2+n}\right)^2 \left(\frac{dx}{d\theta}\right)^2 + \frac{1}{2}m^* x^2. \quad (34)$$

Before going further, we note a difficulty. If $(E/r^n) - (k/n)$ could somehow approach a constant, then Eq. (34) would describe a simple harmonic oscillator of angular frequency

$$\omega = \sqrt{\frac{m^*}{m^*\left(\frac{2}{2+n}\right)^2}} = \frac{2+n}{2} \quad (35)$$

which is identical to the $n < 0$ argument that led to $n = -1$, and therefore contradicts the hypothesis that $n > 0$. Another approach must be found.

Chin found a clever solution around this problem (see Appendix). However, at this point our approach differs from Chin's. Both approaches are correct; we offer one that we find even simpler.

Here is how we see it: With the power-law potential $V \sim r^n$ for $n > 0$, for *small r* (large x) the effective potential is dominated by the $1/r^2$ centrifugal potential, which goes to infinity as $r \to 0$ ($u \to \infty$). For *large r* (small u) the potential energy $V \sim r^n$ dominates, and goes to infinity as $r \to \infty$, i.e., $x \to 0$ (see Fig. 3). The particle is always bound, and r can be made as small or as large as we like if E is sufficiently large. Consider two extreme cases with large E: (a) small r, and (b) large r.

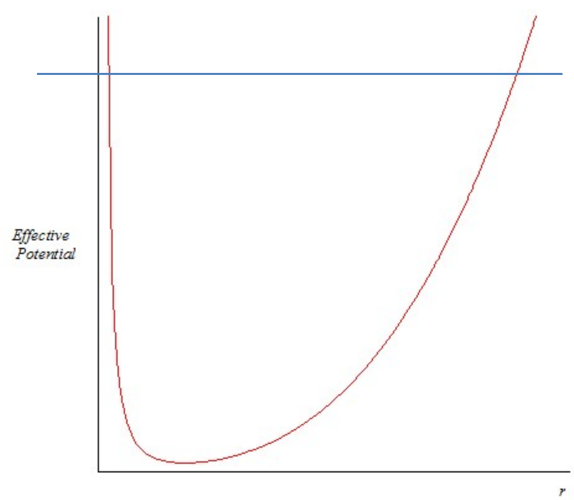

FIGURE 3. The effective potential (red curve) with large E (the horizontal line).

(a) For large E and small r (large u) Eq. (10) becomes

$$E \approx \frac{1}{2}m^*\left(\frac{du}{d\theta}\right)^2 + \frac{1}{2}m^*u^2 \tag{36}$$

which describes a simple harmonic oscillator of angular frequency $\omega_o = \sqrt{m^*/m^*} = 1$. Since $u > 0$, only the positive side of the simple harmonic potential is accessible; thus the frequency is $\omega = 2\omega_o$, and by Eqs. (1) and (32) we have

$$\varphi_A = \frac{\pi}{2} = \frac{\pi}{\sqrt{2+n}} \tag{37}$$

which gives $n = 2$. So far so good, but we must verify that $n = 2$ is consistent with the other extreme.

(b) For large E and large r (small u), the centrifugal potential is negligible, and with $n = 2$,

$$E \approx \frac{1}{2}m^*\left(\frac{du}{d\theta}\right)^2 + \frac{k}{2}r^2. \tag{38}$$

Since $u = 1/r$, by the chain rule and with Eqs. (3) and (10) it follows that

$$\frac{du}{d\theta} = -\frac{m\dot{r}}{L} \tag{39}$$

which restores Eq. (38) back into the original expression for a simple harmonic oscillator subjected to the force $-k\boldsymbol{r}$:

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + \frac{1}{2}kr^2. \tag{40}$$

Evidently, the *only* closed orbit that results when $n > 0$ is $n = +2$.

In summary, for a particle moving in a central potential $V(r) = kr^n$, the orbit will be closed for only two values of n: -1 and $+2$. This is Bertrand's theorem.

APPENDIX

Another Approach When $n > 0$.

In his excellent paper, S. A. Chin[2] took another approach to finding solutions for $n > 0$. Return to Eq. (33) and consider the turning points, where the kinetic energy vanishes. Let u_1 and u_2 be the turning points corresponding to the smallest radius r_1 ($u_1 = 1/r_1$) and the largest radius r_2 ($u_2 = 1/r_2$). With u_k denoting either u_1 or u_2, at the turning points the total energy is entirely carried by the effective potential, so that

$$E = V_e(u_k^{-1}) = \frac{1}{2}m^*u_k^2 + \frac{k}{n}u_k^{-n} \tag{41}$$

where $n > 0$. The orbit is bound, so as $E \to \infty$, r can become very small (u very large). For large E and large u the centrifugal potential dominates, and so

$$E \approx \frac{1}{2}m^*u_1^2 \equiv E_1. \tag{42}$$

Let us now form the ratio

$$\Lambda(u) \equiv \frac{V_e(u^{-1})}{E_1} \tag{43}$$

which by Eqs. (41)-(42) becomes

$$\Lambda(u) = \frac{u^2}{u_1^2} + \frac{ku^{-n}}{nE_1}. \tag{44}$$

Let $x \equiv u/u_1$. Now Eq. (44) can be rearranged into the form

$$\Lambda(x) = x^2 + \frac{k}{n}\left(\frac{m^*}{2}\right)^{n/2} \frac{x^{-n}}{E_1^{1+n/2}}. \tag{45}$$

As $E_1 \to \infty$, $\Lambda(x) \approx x^2$, or $V_e \approx E_1 x^2$. Return this to Eq. (33), which becomes

$$E \approx \frac{1}{2} m^* \left(\frac{du}{d\theta}\right)^2 + E_1 x^2. \tag{46}$$

Noting that $u = u_1 x$, it follows that

$$\frac{du}{d\theta} = \sqrt{\frac{2E_1}{m^*}} \frac{dx}{d\theta} \tag{47}$$

and thus

$$E \approx E_1 \left[\left(\frac{dx}{d\theta}\right)^2 + x^2\right]. \tag{48}$$

This can be rearranged to resemble the energy of a simple harmonic oscillator:

$$\frac{E}{2E_1} = \frac{1}{2}\left(\frac{dx}{d\theta}\right)^2 + \frac{1}{2}x^2 \tag{49}$$

which has angular frequency $\omega_o = 1$, and again since only the positive half of the harmonic oscillator potential can be used, one obtains $\omega = 2$, and thus from Eqs. (13) and (32), $n = 2$.

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